

# An Efficient Method to Calculate the Failure Rate of Dynamic Systems with Random Parameters using the Total Probability Theorem

**Monica Majcher, Zissimos P. Mourelatos, Vasileios Geroulas**  
Mechanical Engineering Department, Oakland University, Rochester MI 48309  
**Igor Baseski, Amandeep Singh**  
U.S. Army TARDEC, Warren MI 48397

## Abstract

Using the total probability theorem, we propose a method to calculate the failure rate of a linear vibratory system with random parameters excited by stationary Gaussian processes. The response of such a system is non-stationary because of the randomness of the input parameters. A space-filling design, such as optimal symmetric Latin hypercube sampling or maximin, is first used to sample the input parameter space. For each design point, the output process is stationary and Gaussian. We present two approaches to calculate the corresponding conditional probability of failure. A Kriging metamodel is then created between the input parameters and the output conditional probabilities allowing us to estimate the conditional probabilities for any set of input parameters. The total probability theorem is finally applied to calculate the time-dependent probability of failure and the failure rate of the dynamic system. The proposed method is demonstrated using a vibratory system. Our approach can be easily extended to non-stationary Gaussian input processes.

## Introduction

The response of a vibratory system with random parameters excited by stationary Gaussian processes is a non-stationary random process. A time-dependent reliability analysis is thus, needed to calculate the probability that the system will perform its intended function successfully for a specified time.

Reliability is an important engineering requirement for consistently delivering acceptable product performance through time. As time progresses, the product may fail due to time-dependent operating conditions and material properties, component degradation, etc. The reliability degradation with time may increase the lifecycle cost due to potential warranty costs, repairs and loss of market share. In this article, we use time-dependent reliability concepts associated with the first-passage of non-repairable systems. Among its many applications, the time-dependent reliability concept can be used to reduce the lifecycle cost [1] or to set a schedule for preventive condition-based maintenance [2].

The time-dependent probability of failure, or cumulative probability of failure [1, 3], is defined as

$$P_f(0, T) = P\{\exists t \in [0, T]: g(\mathbf{X}, \mathbf{F}(t), t) \leq 0\} \quad (1)$$

where the limit state  $g(\mathbf{X}, \mathbf{Z}(t), t) = 0$  depends on the vector  $\mathbf{X} = [X_1 \ X_2 \ \dots \ X_m]$  of  $m$  input random variables, the vector  $\mathbf{F}(t) = [F_1(t) \ F_2(t) \ \dots \ F_n(t)]$  of  $n$  input random processes. Failure occurs if  $g(\cdot) \leq 0$  at any time  $t \in [0, T]$  where  $T$  is the time of interest.

The time-dependent probability of failure of Eq. (1) can be calculated exactly as

$$P_f(0, t) = 1 - \left(1 - P_f^0\right) \exp\left\{-\int_0^t \lambda(\xi) d\xi\right\} \quad (2)$$

where  $P_f^0$  is the instantaneous probability of failure at the initial time and  $\lambda(t)$  is the failure rate. Eq. (2) indicates that  $P_f(0, T)$  can be calculated if  $\lambda(t)$  is known and vice versa.

In the commonly used up-crossing rate approach, the failure rate is approximated by the up-crossing rate

$$\nu^+(t) = \lim_{\Delta t \rightarrow 0} \frac{P[g(\bullet, t) > 0 \cap g(\bullet, t + \Delta t) \leq 0]}{\Delta t} \quad (3)$$

under the assumptions that the probability of having two or more out-crossings in  $[t, t + \Delta t]$  is negligible, and the out-crossings in  $[t, t + \Delta t]$  are statistically independent of the previous out-crossings in  $[0, t]$ . Eq. (2) is then used to estimate  $P_f(0, T)$ .

Monte Carlo simulation (MCS) can accurately estimate the probability of failure of Eq. (1) but it is computationally prohibitive for low failure of probability problems. To address the computational issue of MCS, analytical methods have been developed based on the out-crossing rate approach which was first introduced by Rice [4] followed by extensive studies [3, 5-7]. The PHI2 method [3] uses two

Report Documentation Page			Form Approved OMB No. 0704-0188		
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE <b>12 MAY 2015</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-2015 to 00-00-2015</b>	
4. TITLE AND SUBTITLE <b>An Efficient Method to Calculate the Failure Rate of Dynamic Systems with Random Parameters using the Total Probability Theorem</b>			5a. CONTRACT NUMBER		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S)			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>US Army RDECOM-TARDEC,6501 E. 11 Mile Road,Warren,MI,48397-5000</b>			8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S)		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S)		
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT <b>See Report</b>					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT <b>Same as Report (SAR)</b>	18. NUMBER OF PAGES <b>9</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			

successive time-invariant analyses based on FORM, and the binomial cumulative distribution to calculate the probability of the joint event in Eq. (3). A Monte-Carlo based set theory approach has been also proposed [8] using a similar approach with the PHI2 method. Analytical studies such as in [9, 10] have shown that the PHI2-based approach lacks sufficient accuracy for vibratory systems. Other analytical approaches have been however, proposed to estimate the time-dependent probability of failure with sufficient accuracy [11, 12].

The limited accuracy of the out-crossing rate approach has been improved by solving an integral equation involving  $\nu^+(t)$  and  $\nu^{++}(t, t_1)$ , the joint up-crossing rate between times  $t$  and  $t_1$  [13]. The up-crossing rate  $\nu^+(t)$  is defined in Eq. (3) and the joint up-crossing rate  $\nu^{++}(t, t_1)$  is defined as

$$\nu^{++}(t, t_1) = \lim_{\Delta t \rightarrow 0} \frac{P[g(\bullet, t) > 0 \cap g(\bullet, t + \Delta t) \leq 0 \cap g(\bullet, t_1) > 0 \cap g(\bullet, t_1 + \Delta t) \leq 0]}{(\Delta t)^2} \quad (4)$$

This approach has been adopted in [10].

Among the simulation-based methods, a MCS approach was proposed in [14] to estimate the time-dependent failure rate over the product lifecycle and its efficiency was improved using an importance sampling method considering a decorrelation length [15] in order to reduce the high dimensionality of the problem. Subset simulation [16] has been also developed as an efficient simulation method for computing small failure probabilities for general reliability problems. Its efficiency comes from introducing appropriate intermediate failure sub-domains to express the low probability of failure as a product of larger conditional failure probabilities which are estimated with much less computational effort. An extreme value method has also been proposed [17] using the distribution of the extreme value of the response. Recently, a time-dependent reliability analysis was proposed [18] using the total probability theorem and the concept of composite limit state.

In this paper, we present a time-dependent reliability analysis for dynamic systems with random parameters excited by a stationary Gaussian process using the total probability theorem. Metamodels are used to estimate conditional probabilities needed by the total probability theorem. An advantage of our approach is that we can easily handle non-normal and correlated random variables without additional computational effort.

The paper is arranged as follows. Section 2 describes the proposed methodology with all necessary details. Section 3 uses a beam example to demonstrate all developments and Section 4 summarizes, concludes and highlights future research.

## Proposed Approach

We consider the following  $n$  degree-of-freedom (DOF) linear vibratory system with random parameters

$$[m(\mathbf{X})]\ddot{\mathbf{Y}}(t) + [c(\mathbf{X})]\dot{\mathbf{Y}}(t) + [k(\mathbf{X})]\mathbf{Y}(t) = \mathbf{F}(t) \quad (5)$$

where  $\mathbf{Y}(t) = [Y_1(t) \ Y_2(t) \ \dots \ Y_n(t)]$  is the vector of output (response) random processes,  $\mathbf{F}(t) = [F_1(t) \ F_2(t) \ \dots \ F_n(t)]$  is the vector of input (force) random processes, and  $\mathbf{X} = [X_1 \ X_2 \ \dots \ X_m]$  includes  $m$  input random variables (see Figure 1). The mass  $[m]$ , damping  $[c]$  and stiffness  $[k]$  matrices depend on  $\mathbf{X}$ . Although our approach can handle all  $n$  random processes in  $\mathbf{Y}(t)$  and  $\mathbf{F}(t)$ , we will consider the case with only one input  $F_i(t)$ ,  $1 \leq i \leq n$  and one output  $Y_j(t)$ ,  $1 \leq j \leq n$  for simplicity. Figure 1 provides a schematic with the input-output notation.



Figure 1. Schematic of linear vibratory system.

## Total Probability Theorem Approach

The calculation of the time-dependent probability of failure (i.e., the probability of the response exceeding a threshold) is very challenging because of the random parameters  $\mathbf{X}$ . For each realization of  $\mathbf{X}$  however, the linear vibratory system of Eq. (5) has constant coefficients (matrices  $[m]$ ,  $[c]$  and  $[k]$ ). The problem then becomes much easier if the system is excited by wide-sense stationary processes  $\mathbf{F}(t)$ . The total probability theorem allows us to calculate the  $P_f(0, t)$  of the dynamic system with random parameters in terms of conditional probabilities of dynamic systems with constant parameters.

According to the total probability theorem, the time-dependent probability of failure of Eq. (1) can be expressed as [18]

$$P_f(0, T) = P(F) = \int_{\Omega} P\left(\frac{F}{\mathbf{X}}\right) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (6)$$

where  $P(F/\mathbf{X})$  is a time-dependent conditional probability of failure,  $f_{\mathbf{X}}(\mathbf{x})$  is the joint PDF of the input random variables  $\mathbf{X}$  and  $\Omega$  is the support of  $f_{\mathbf{X}}(\mathbf{x})$ . The integral of Eq. (6) can be calculated using numerical integration schemes if the number  $n$  of random variables is small (e.g. less than 5). Otherwise, Monte Carlo simulation or importance sampling methods can be used. In all cases,  $P(F/\mathbf{X})$  is calculated directly or using a pre-built metamodel as is the case in this paper. An advantage of the total probability theorem is that non-normal and correlated random variables are handled without additional computational effort or loss of accuracy.

## Calculation of Conditional Probability $P(F/\mathbf{X})$

We assume that the input random process  $F(t)$  is a wide-sense stationary and Gaussian random process with zero mean and a

constant standard deviation, characterized fully by the power spectral density (PSD)  $S_{FF}(\omega)$ . In this case, for *each realization* of the input random variables  $\mathbf{X}$ , the output process  $Y(t)$  is also wide-sense stationary and Gaussian with zero mean, a constant standard deviation and a PSD of

$$S_{YY}(\omega) = |H(\omega)|^2 S_{FF}(\omega) \quad (7)$$

where  $H(\omega)$  is the frequency response function of the system between the  $i^{th}$  (input) and  $j^{th}$  (output) degrees of freedom (DOF) [19].

For a real-life, multi-degree of freedom system with millions of DOF, the calculation of  $H(\omega)$  is computationally intensive. It is desired therefore to minimize the number of times  $H(\omega)$  is calculated. Our approach using the total probability theorem reduces the number  $H(\omega)$  is calculated considerably.

According to the Wiener-Khintchine theorem, the Fourier transform of the  $S_{YY}(\omega)$  spectrum provides the autocorrelation function  $R_{YY}(\tau)$  of the output process. Because both  $S_{YY}(\omega)$  and  $R_{YY}(\tau)$  are real and even functions, we have [19]

$$R_{YY}(\tau) = \frac{1}{\pi} \int_0^{+\infty} S_{YY}(\omega) \cos(\omega\tau) d\omega. \quad (8)$$

Note that for a wide-sense stationary process,

$$R_{YY}(\tau) = E[Y(t) \cdot Y(t+\tau)] = \rho\sigma_Y^2 + \mu_Y^2 \quad (9)$$

where  $-1 \leq \rho \leq +1$  is the correlation coefficient and  $E[Y(t)] = \mu_Y$  and  $\sigma_{Y(t)} = \sigma_Y$ . The covariance function is

$$\begin{aligned} C_{YY}(t, t+\tau) &= R_{YY}(t, t+\tau) - E[Y(t)] \cdot E[Y(t+\tau)] = \\ &= C_{YY}(\tau) = R_{YY}(\tau) - \mu_Y^2 \end{aligned} \quad (10)$$

Knowing the covariance function of the output process, we can use a spectral decomposition method to express the output process as a linear combination of the eigenvectors of the covariance matrix where the coefficients are independent standard normal random variables. The time interval of interest  $[0, T]$  is discretized using  $N$  discrete times  $t_1=0, t_2, \dots, t_N=T$  and the  $N \times N$  covariance matrix  $\Sigma = [Cov(t_i, t_j)]$ ,  $i=1, 2, \dots, N; j=1, 2, \dots, N$  is formed where  $Cov(t_i, t_j) = C_{YY}(t_i, t_j) = R_{YY}(\tau)$  is the covariance between times  $t_i$  and  $t_j$  provided by Eq. (10).

Let  $\Sigma = \Phi \cdot \Lambda \cdot \Phi^T$  be the spectral decomposition of the covariance matrix  $\Sigma$  where  $\Phi = [\Phi_1 \ \Phi_2 \ \dots \ \Phi_N]$  is the

orthonormal matrix of the eigenvectors  $\Phi_i, i=1, \dots, N$  and  $\Lambda = diag[\lambda_1 \ \lambda_2 \ \dots \ \lambda_N]$  is the diagonal matrix of the corresponding eigenvalues. Also, let  $\mathbf{Z} = (Z_1 \ Z_2 \ \dots \ Z_N)$  be a vector of  $N$  independent standard normal variables. Because of the affine transformation property of the multi-normal distribution, the following spectral representation holds

$$Y(t) = \mu_Y(t) + \sum_{i=1}^r \sqrt{\lambda_i} \cdot \Phi_i^T(t) \cdot Z_i \quad (11)$$

where  $t = t_1, t_2, \dots, t_N$  and  $r \leq N$  is the number of dominant eigenvalues. Eq. (6) can be used to generate sample functions (trajectories) of  $Y(t)$ . The Expansion Optimal Linear Estimation method (EOLE) or the Orthogonal Series Expansion (OSE) [20] can also be used.

Since Eq. (11) provides a discretized version of the output process, we can use Monte Carlo simulation to calculate the time-dependent probability of failure. However, MCS will be computationally very demanding considering that the time of interest  $T$  can be long and the time step  $\Delta t$  very small. To reduce the computational effort, we use in this paper the total probability theorem.

We use two approaches to estimate  $P(F/\mathbf{X})$ . The first one assumes that the up-crossings are statistically independent while the second approach does not.

### Approach 1: Statistically Independent Up-crossings

The  $P(F/\mathbf{X}) = P_f[0, T]$  is estimated using Eq. (2) where the failure rate  $\lambda(t)$  is approximated by the up-crossing rate  $v^+(t)$  (i.e.,  $\lambda(t) = v^+(t)$ ). Because the excitation process is Gaussian and wide-sense stationary and the dynamic system is linear with constant parameters, the up-crossing rate of threshold  $\alpha$  is constant given by [19]

$$v^+(t) = v^+ = v_0^+ e^{-0.5 \left( \frac{\alpha}{\sigma_Y} \right)^2} \quad (12)$$

where

$$v_0^+ = \frac{\sigma_{\dot{Y}}}{2\pi\sigma_Y} \quad (13)$$

is the up-crossing rate for  $\alpha = 0$  and

$$\sigma_{\dot{Y}} = \sqrt{\int_{-\infty}^{\infty} \omega^2 S_{YY}(\omega) d\omega} \quad (14)$$

is the standard deviation of the derivative process  $\dot{Y}(t)$ .

**Approach 2: Statistically Dependent Up-Crossings**

This approach demonstrates the significance of considering the correlation among up-crossings in  $[0, T]$ . We define the failure event as  $F = \{\exists t \in [0, T]: g(\mathbf{X}, \mathbf{F}(t), t) = \alpha - Y(t) \leq 0\}$ . The probability  $P_f[0, t]$ ,  $0 \leq t \leq T$  is then given by

$$P_f[0, t] = P_f^0 + (1 - P_f^0) \int_0^t f(\tau) d\tau \quad (15)$$

where  $f(t)$  is the PDF of the first time to failure and

$$P_f^0 = P\{Y(0) > \alpha\} \quad (16)$$

is the instantaneous probability of failure at  $t = 0$ .

The up-crossing rate at time  $t$  is the probability that the first up-crossing occurred at  $t$  or at a previous time  $\tau$ . This is expressed as [13, 10]

$$v^+(t) = f(t) + \int_0^t v^+(t | \tau) f(\tau) d\tau \quad (17)$$

where  $v^+(t | \tau)$  is the up-crossing rate at  $t$  conditioned on the time of the first up-crossing being  $\tau$ . Based on Eq. (17),  $f(t)$  has the following upper bound

$$f(t) \leq v^+(t) \quad (18)$$

because the integral is a non-negative quantity.

The joint up-crossing rate  $v^{++}(t, \tau_1)$ , indicating the probability that an up-crossing occurs at both  $t$  and  $\tau_1 < t$ , can be expressed as

$$v^{++}(t, \tau_1) = v^+(t | \tau_1) f(t) + \int_0^{\tau_1} v^+(t, \tau_1 | \tau_2) f(\tau_2) d\tau_2 \quad (19)$$

since the up-crossing at  $\tau_1$  is the first or the first up-crossing has occurred at some previous time  $\tau_2 < \tau_1$ . Integration of Eq. (19) and substitution in Eq. (17) yields,

$$\begin{aligned} f(t) = v^+(t) - \int_0^t v^{++}(t, \tau_1) d\tau_1 + \\ + \int_0^t \int_0^{\tau_1} v^+(t, \tau_1 | \tau_2) f(\tau_2) d\tau_2 d\tau_1 \end{aligned} \quad (20)$$

Because the integral in Eq. (20) is non-negative,  $f(t)$  has the following lower bound

$$f(t) \geq v^+(t) - \int_0^t v^{++}(t, \tau_1) d\tau_1. \quad (21)$$

If we continue the process of using Equations (17) and (19) to derive Eq. (20), the following Rice's inclusion-exclusion series [4, 19] can be obtained

$$f(t) = v^+(t) + \sum_{i=2}^{\infty} (-1)^{i-1} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{i-1}} v^{i+}(t, \tau_1, \dots, \tau_{i-1}) d\tau_1 \cdots d\tau_{i-1} \quad (22)$$

where

$$v^{i+}(\cdot) = \begin{cases} v^{++}(\cdot) & \text{if } i = 2 \\ v^{+++}(\cdot) & \text{if } i = 3 \\ \vdots & \end{cases} \quad (23)$$

The PDF  $f(t)$  is then used in Eq. (15) to obtain the time-dependent probability of failure  $P_f(0, t)$ .

For the series of Eq. (22) to converge, we may need many terms; i.e., joint up-crossing rates of higher order (large  $i$ ). To avoid this, the integral Eq. (17) can be modified as

$$v^+(t) = f(t) + \int_0^t \frac{v^{++}(t, \tau)}{v^+(\tau)} f(\tau) d\tau \quad (24)$$

where the conditional probability definition is used to replace the

conditional up-crossing rate  $v^+(t | \tau)$  with  $\frac{v^{++}(t, \tau)}{v^+(\tau)}$  where

$v^+(\tau)$  is the up-crossing rate for the *first* up-crossing. Because the up-crossing rate for the *first* up-crossing is not known, an approximation is obtained if  $v^+(\tau)$  is the up-crossing rate for *any* up-crossing at  $\tau < t$ . It was shown in [10, 13] that this approximation provides accurate results. The example in this paper verifies this claim.

In our Approach 2, we solve the integral Eq. (24) for  $f(t)$  and then use it in Eq. (15) to obtain the time-dependent probability of failure  $P_f(0, t)$ .

Note that if all up-crossings are assumed statistically independent, the high level joint up-crossing rate  $v^{i+}(t, \tau_1, \dots, \tau_{i-1})$  in Eq. (22) is equal to

$$v^{i+}(t, \tau_1, \dots, \tau_{i-1}) = v^+(t) \cdot v^+(\tau_1) \cdots v^+(\tau_{i-1}). \quad (25a)$$

If in addition, the process is stationary, the up-crossing rate does not depend on time and

$$v^{i+}(t, \tau_1, \dots, \tau_{i-1}) = v^+(t) \cdot v^+(\tau_1) \cdots v^+(\tau_{i-1}) = (v^+)^i. \quad (25b)$$

In this case, the Rice's series of Eq. (22) yields

$$f(t) = v^+ - (v^+)^2 t + (v^+)^3 \frac{t^2}{2} - (v^+)^4 \frac{t^3}{6} + \dots = v^+ e^{-v^+ t}. \quad (26)$$

Therefore, the failure rate

$$\lambda(t) = \frac{f(t)}{1 - \int_0^t f(t) dt} = \frac{v^+ e^{-v^+ t}}{1 + [e^{-v^+ t} - e^0]} = \frac{v^+ e^{-v^+ t}}{e^{-v^+ t}} = v^+ \quad (27)$$

is constant and equal to the up-crossing rate  $v^+$ .

### Calculation of $v^{++}(t, \tau)$ for Approach 2

In Eq. (24), we need the up-crossing rate  $v^+(\tau)$  and the joint up-crossing rate  $v^{++}(t, \tau)$ . The former is constant for each realization of  $\mathbf{X}$  and is calculated using Eq. (12). We use the following steps to calculate  $v^{++}(t, \tau)$ :

- Based on Eq. (3),  $P(S_t \cap F_{t+\Delta t}) = v^+ \Delta t$  where  $S_t$  indicates the safe event at  $t$  and  $F_{t+\Delta t}$  indicates the failure event at  $t + \Delta t$ .
- Calculate an equivalent reliability index  $\beta = -\Phi^{-1}(v^+ \Delta t)$ . Because of stationarity at a realization of  $\mathbf{X}$ ,  $\beta$  is the same for event  $(S_\tau \cap F_{\tau+\Delta t})$ .
- Calculate the intersection probability  $P_{t,\tau}$  using the bivariate standard normal CDF  $\Phi_2$  as [18]
$$P_{t,\tau} = \Phi_2(-\beta, -\beta, \rho) = \frac{1}{2\pi} \int_{\arccos(\rho)}^{\pi} \exp\left[-\frac{1}{2}(\beta^2 + \beta^2 - 2\beta\beta \cos\theta)\csc^2\theta\right] d\theta$$
 where  $\beta$  is obtained from the previous step and  $\rho$  is the correlation coefficient between the linear safety margins at  $t$  and  $\tau$ .
- Based on the definition of Eq. (4), we have
$$v^{++}(t, \tau) = P_{t,\tau} (\Delta t)^2.$$

### Vibratory Beam Example

We demonstrate the proposed approach using the single degree of freedom system of Figure 2 adopted from [21]. A concentrated mass  $M$  is placed at the mid-span of a massless beam of length  $L = 4m$ . A random load  $F(t)$  is applied on the mass which deflects by  $y(t)$  as shown in the figure. The beam is made of steel with Young modulus  $E = 210GPa$  and density  $\rho = 7850 \text{ Kg/m}^3$ . The beam has a rectangular cross section of width  $b$  and height  $h$  and provides a stiffness  $K = \frac{6EI}{L^3}$  to the mass-damper system with  $I = \frac{bh^3}{12}$  being the area moment of inertia. There is also a damper  $C$  attached to the mass.

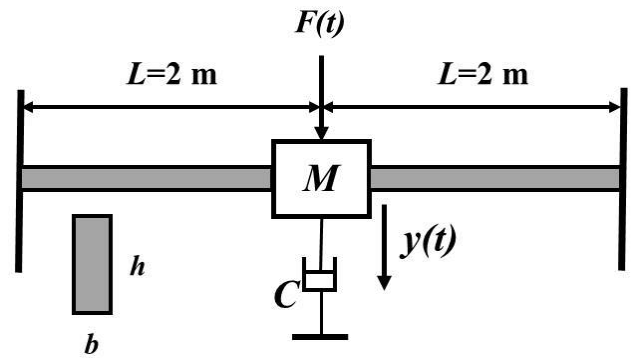


Figure 2 . Beam under random loading [21]

The parameters  $M$ ,  $b$  and  $h$  are normally distributed with  $\mu_M = 100kg$ ,  $\mu_b = 0.01m$ ,  $\mu_h = 0.05m$  and  $\sigma_M = 5kg$ ,  $\sigma_b = 0.001m$ ,  $\sigma_h = 0.005m$ . The random load  $F(t)$  is a zero mean, wide-sense stationary process with the following Pierson-Moskowitz spectrum

$$S_{FF}(\omega) = \frac{A}{\omega^5} e^{\frac{-B}{\omega^4}} \quad (28)$$

where  $A = 6.5 \cdot 10^9 \text{ N}^2 \text{ sec}^{-4}$  and  $B = 36,219 \text{ sec}^{-4}$  (see Figure 3).

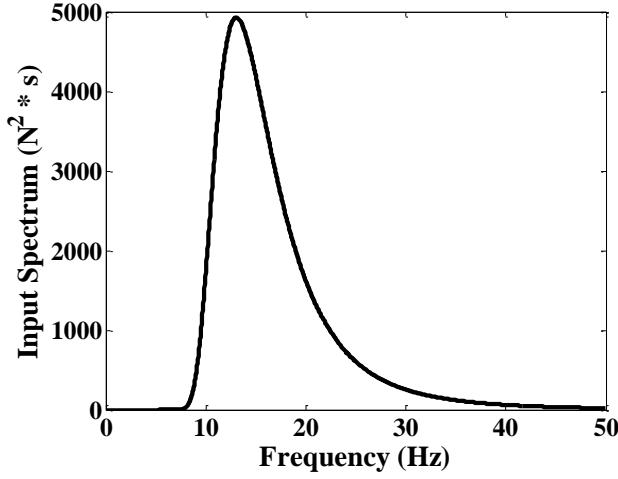


Figure 3. Input force spectrum

The equation of motion for the system is

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) = F(t). \quad (29)$$

The undamped natural frequency is  $\omega_n = \sqrt{\frac{6EI}{ML^3}}$ . A damping ratio  $\zeta = 0.3$  is used resulting in a damping value of  $C = 2M\omega_n\zeta$ . The mean value of the undamped natural frequency is 12.8 rad/sec. The failure event is defined as

$$F = \{\exists t \in [0, T]: g(\mathbf{X}, \mathbf{F}(t), t) = 0.02 - Y(t) \leq 0\}$$

indicating that we have failure if the response  $y(t)$  exceeds the threshold  $\alpha = 0.02 m$ .

For the linear, one degree of freedom vibratory system of Eq. (29), the transfer function is

$$H(\omega) = \frac{Y(\omega)}{F(\omega)} = \frac{1}{-M\omega^2 + jC\omega + K} \quad (30)$$

and the output spectral density is

$$S_{YY}(\omega) = |H(\omega)|^2 S_{FF}(\omega) = \frac{S_{FF}(\omega)}{(K - M\omega^2)^2 + C^2\omega^2}. \quad (31)$$

Figure 4 compares the conditional time-dependent probability of failure  $P(F/\mathbf{X})$  for  $M = 100kg$ ,  $b = 0.01 m$  and  $h = 0.05m$  (mean value point of the three random variables) among Approach 1, Approach 2 and MC simulation. The MC results have been obtained with 20,000,000 replications. As expected, assuming that the up-crossings are statistically independent and using Eq. (2) with  $\lambda(t) = v^+$  to calculate the  $P(F/\mathbf{X})$ , overestimates the actual probability. However the results of Approach 2, using Eq. (24) and Eq. (15), are very close to MCS.

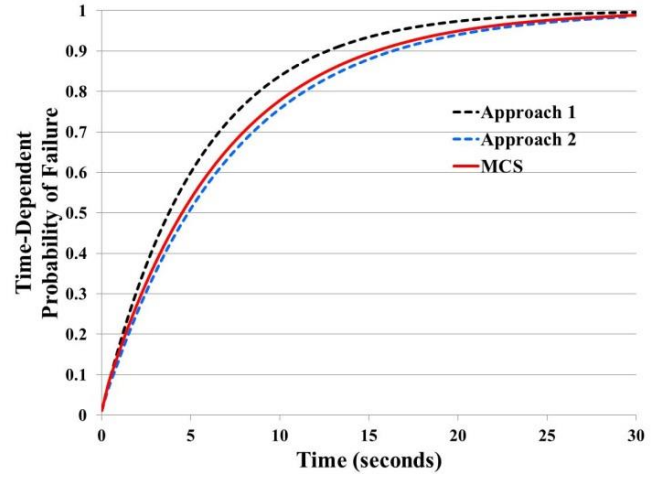


Figure 4. Comparison of conditional time-dependent probability of failure  $P(F/\mathbf{X})$  for  $M=100 kg$ ,  $b=0.01 m$  and  $h=0.05 m$

### Approach 1

We first developed a Kriging metamodel for  $v^+$  (Eq. 12) in order to avoid calculating repeatedly the expensive transfer function  $H(\omega)$  needed to calculate the output spectrum  $S_{YY}(\omega)$  in Eq. (14). Note that although  $H(\omega)$  is not computationally expensive in this example, it is very expensive for large-scale vibratory systems with many DOF. We also developed another Kriging metamodel for  $P_f^0$  in Eq. (2). According to the total probability theorem, the time-dependent  $P_f[0, T]$  of the system is the mean value of the conditional  $P(F/\mathbf{X})$ . We generated 10,000 replications of  $v^+$  and  $P_f^0$  from the metamodels, and Eq. (2) was used for each replication to calculate  $P(F/\mathbf{X})$  using  $\lambda(t) \approx v^+$ .

For the Kriging metamodels, we used an Optimal Symmetric Latin Hypercube (OSLH) design [22] to sample the design space. The range of values for each random variable was  $[\mu - 3\sigma, \mu + 3\sigma]$ . We started with 20 design points, and increased the number of points in increments of 10 until convergence of  $P_f[0, T = 30]$  was reached. The “leave-one-out” validation procedure was also used for a few points in the design space to check the accuracy of the Kriging metamodels. Thirty design points were needed for convergence (see Figure 5). The DACE Matlab Toolbox was used to build the Kriging metamodels. A second-order, Gaussian correlation structure was used for the metamodel of  $v^+$ , and a first-order, Gaussian correlation structure was used for the metamodel of  $P_f^0$ .



**Approach 2**

The total probability theorem of Eq. (6) was used. To determine the time-dependent conditional probability  $P(F/\mathbf{X})$  for a realization  $\mathbf{X} = \{M \ b \ h\}$  of the random variables, we first solved the integral Eq. (24) numerically for the PDF  $f(t)$  of the first time to failure for  $0 \leq t \leq 30$  using a  $\Delta t = 0.4 \text{ sec}$  time step. Subsequently, Eq. (15) was used to calculate  $P(F/\mathbf{X}) = P_f[0, t], 0 \leq t \leq T$ .

The  $P(F/\mathbf{X})$  curve was calculated for each point of an OSLH design similarly to Approach 1. The same 30 OSLH points were used. Figure 8 shows all curves. A time-dependent metamodel [23] was then developed using a singular value decomposition method.

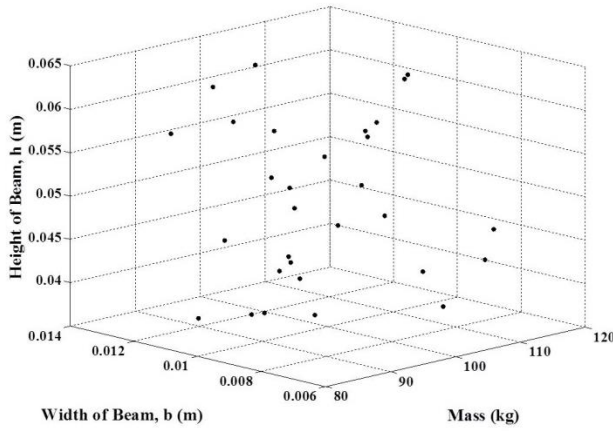


Figure 5. Design of 30 OSLH points

Figures 6 and 7 show the output spectra and the corresponding autocorrelation functions for the 30 OSLH design points.

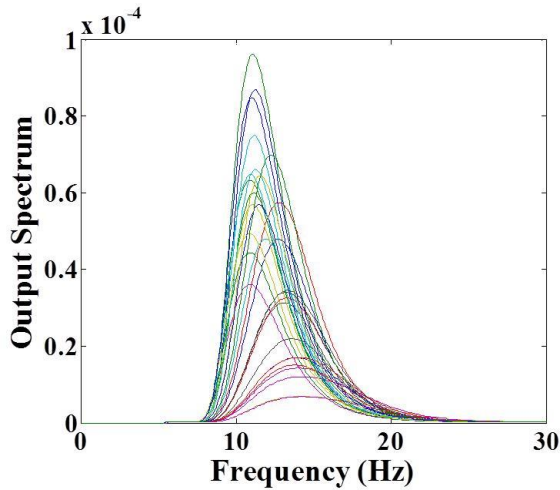


Figure 6. Output spectra for the 30 OSLH points

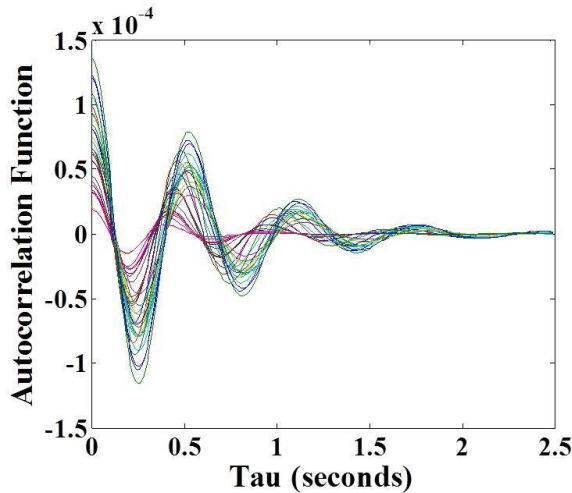


Figure 7. Autocorrelation function for the 30 OSLH points

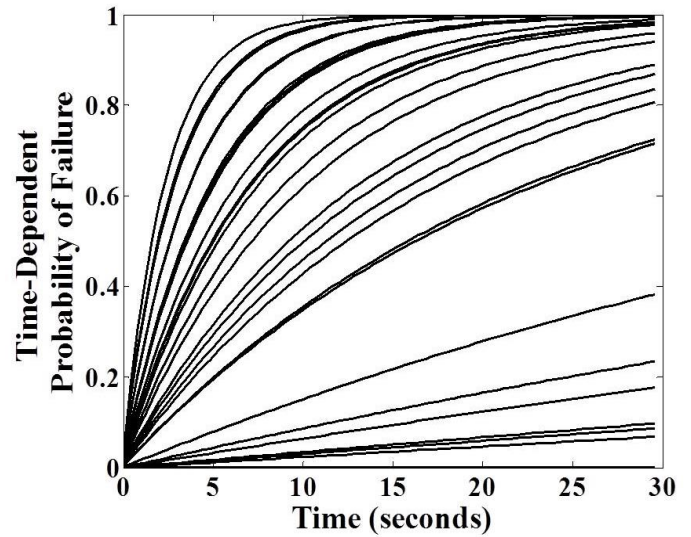
Figure 8.  $P(F/\mathbf{X})$  curves for the 30 OSLH points for Approach 2

Figure 9 compares the time-dependent probability of failure between the two approaches. Because of the statistically independent assumption of the up-crossings, the Approach 1 overestimates the probability of failure. The results from Approach 2 are much more accurate as shown in Figure 4 where Approach 2 is much closer to MCS for the case where all random variables are at their means. Figure 10 compares the time-dependent failure rate calculated using the two approaches. Figure 11 shows the joint up-crossing rate  $v^{++}(t, \tau)$  for the design point with all random variables at their mean values.



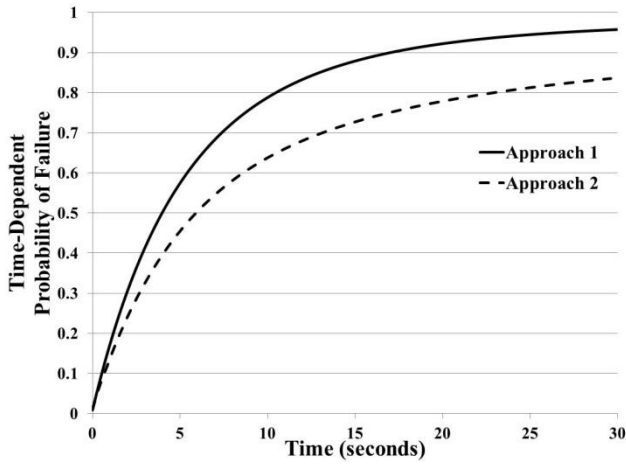


Figure 9. Comparison of time-dependent probability of failure

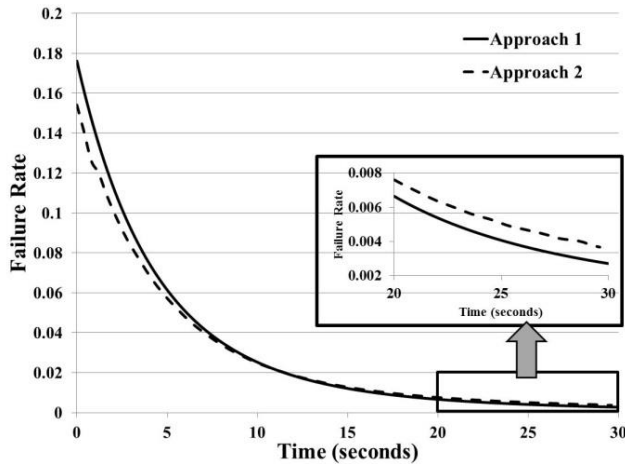


Figure 10. Comparison of time-dependent failure rate

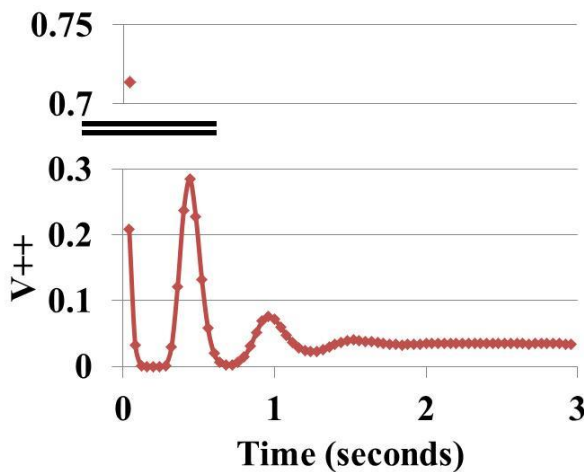


Figure 11. Joint up-crossing rate for design point with all random variables at their means

## Summary, Conclusions and Future Work

We presented a methodology to calculate the time-dependent probability of failure and the failure rate of a linear vibratory system with random parameters excited by stationary Gaussian processes using the total probability theorem and an integral equation involving the up-crossing and joint up-crossing rates. An optimal symmetric Latin hypercube space-filling design is first used to sample the system random parameters. Time-dependent conditional probabilities are then calculated at each design point by solving an integral equation involving up-crossing and joint up-crossing rates as well as the PDF of the first time to failure. A time-dependent metamodel of the conditional probabilities is built and used in the total probability theorem to calculate efficiently and accurately the time-dependent probability of failure and the failure rate of the vibratory system.

Our approach assumes a wide-sense stationary and Gaussian excitation. However, it can be easily extended to handle non-stationary Gaussian excitations. An example of a simple vibratory system was used to demonstrate all developments.

## Acknowledgment

We would like to acknowledge the technical and financial support of the Automotive Research Center (ARC) in accordance with Cooperative Agreement W56HZV-04-2-0001 U.S. Army Tank Automotive Research, Development and Engineering Center (TARDEC) Warren, MI.

## References

1. Singh, A., Mourelatos, Z. P., and Li, J., 2010, "Design for Lifecycle Cost using Time-dependent Reliability," *ASME Journal of Mechanical Design*, 132(9), 091008 (11 pages).
2. Singh, A., Mourelatos, Z. P., and Li, J., 2010, "Design for Lifecycle Cost and Preventive Maintenance Using Time-Dependent Reliability," *Advanced Materials Research*, 118-120, pp. 10-16.
3. Andrieu-Renaud, C., Sudret, B., and Lemaire, M., 2004, "The PHI2 Method: A Way to Compute Time-Variant Reliability," *Reliability Engineering and Safety System*, 84(1), 75-86.
4. Rice, S. O., 1954, "Mathematical Analysis of Random Noise," *Bell Syst Tech J*, 23, 282-332. [Re-published in: Wax N, editor. *Selected Papers on Noise and Stochastic Processes*, New York, Dover, 1954].
5. Rackwitz, R., 1998, "Computational Techniques in Stationary and Non-Stationary Load Combination – A Review and Some Extensions," *Journal of Structural Engineering*, 25(1), 1-20.
6. Sudret, B., 2008, "Analytical Derivation of the Outcrossing Rate in Time-variant Reliability Problems," *Structure and Infrastructure Engineering*, 4(5), 356-362.
7. Zhang, J., and Du, X., 2011, "Time-dependent Reliability Analysis for Function Generator Mechanisms," *ASME Journal of Mechanical Design*, 133(3), 031005 (9 pages).
8. Savage, G. J., and Son, Y. K., 2009, "Dependability-Based Design Optimization of Degrading Engineering Systems," *ASME Journal of Mechanical Design*, 131(1), 011002 (10 pages).
9. Singh, A., and Mourelatos, Z. P., 2010, "On the Time-dependent Reliability of Non-Monotonic, Non-Repairable Systems," *SAE*

*International Journal of Materials and Manufacturing*, 3(1), 425-444.

10. Hu, Z., Li, H., Du, X., and Chandrashekhara, K., 2012, "Simulation-based Time-dependent Reliability Analysis for Composite Hydrokinetic Turbine Blades," *Structural and Multidisciplinary Optimization*, DOI 10.1007/s00158-012-0839-8.
11. Li, J., and Mourelatos, Z. P., 2009, "Time-Dependent Reliability Estimation for Dynamic Problems using a Niching Genetic Algorithm," *ASME Journal of Mechanical Design*, 131(7).
12. Hu, Z., and Du, X., 2012, "Reliability Analysis for Hydrokinetic Turbine Blades," *Renewable Energy*, 48, 251-262.
13. Madsen, P. H., and Krenk, S., 1984, "An Integral Equation Method for the First Passage Problem in Random Vibration," *Journal of Applied Mechanics*, 51, 674-679.
14. Singh, A., and Mourelatos, Z. P., 2010, "Time-Dependent Reliability Estimation for Dynamic Systems Using a Random Process Approach," *SAE International Journal of Materials & Manufacturing*, 3(1), 339-355.
15. Singh, A., Mourelatos, Z. P., and Nikolaidis, E., 2011, "Time-Dependent Reliability of Random Dynamic Systems using Time-Series Modeling and Importance Sampling," *SAE International Journal of Materials & Manufacturing*, 4(1), 929-946.
16. Au, S. K., and Beck, J. L., 2003, "Subset Simulation and its Application to Seismic Risk Based on Dynamic Analysis," *Journal of Engineering Mechanics*, 129, 901-917.
17. Wang, Z. and Wang, P., 2012, "A Nested Extreme Response Surface Approach for Time-dependent Reliability-based Design Optimization," *ASME Journal of Mechanical Design*, 134(12), 121007 (14 pages).
18. Mourelatos, Z. P., Majcher, M., Pandey, V., and Baseski, I., 2014, "Time-dependent Reliability Analysis Using the Total Probability Theorem," *Proceedings of 2014 ASME IDETC/CIE Conferences*, Paper DETC2014-35078, Buffalo, NY.
19. Nikolaidis, E., Mourelatos, Z. P., and Pandey, V., 2011, *Design Decisions under Uncertainty with Limited Information*, CRC Press, Taylor & Francis Group, London, UK, 525 pages, ISBN 978-0-415-49247-8.
20. Li, C. C., and Kiureghian, A. D., 1993, "Optimal Discretization of Random Fields," *Journal of Engineering Mechanics*, 119(6), 1136-1154.
21. Norouzi, M., and Nikolaidis, E., 2012, "Efficient Method for Reliability Assessment Under High-cycle Fatigue," *International Journal of Reliability, Quality and Safety Engineering*, 19(5), 1250022 (27 pages).
22. Ye, Q., Li, W., and Sudjianto, A., 2003, "Algorithmic Construction of Optimal Symmetric Latin Hypercube Designs," *Journal of Statistical Planning and Inference*, 90, 145-159.
23. Wehrwein D., and Mourelatos, Z. P., 2009, "Optimization of Engine Torque Management under Uncertainty for Vehicle Driveline Clunk using Time-Dependent Metamodels," *ASME Journal of Mechanical Design*, 131(5).

## Contact Information

Zissimos P. Mourelatos, Ph.D.  
 Mechanical Engineering Department  
 Oakland University  
 Rochester, MI 48309  
 Tel: 248-370-2686  
 mourelat@oakland.edu